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## RESEARCH ARTICLE

# Semisimplification for subgroups of reductive algebraic groups

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## Abstract

Let  $G$  be a reductive algebraic group—possibly non-connected—over a field  $k$ , and let  $H$  be a subgroup of  $G$ . If  $G = \mathrm{GL}_n$ , then there is a degeneration process for obtaining from  $H$  a completely reducible subgroup  $H'$  of  $G$ ; one takes a limit of  $H$  along a cocharacter of  $G$  in an appropriate sense. We generalise this idea to arbitrary reductive  $G$  using the notion of  $G$ -complete reducibility and results from geometric invariant theory over non-algebraically closed fields due to the authors and Herpel. Our construction produces a  $G$ -completely reducible subgroup  $H'$  of  $G$ , unique up to  $G(k)$ -conjugacy, which we call a  $k$ -semisimplification of  $H$ . This gives a single unifying construction that extends various special cases in the literature (in particular, it agrees with the usual notion for  $G = \mathrm{GL}_n$  and with Serre's ‘ $G$ -analogue’ of semisimplification for subgroups of  $G(k)$  from [19]). We also show that under some extra hypotheses, one can pick  $H'$  in a more canonical way using the Tits Centre Conjecture for spherical buildings and/or the theory of optimal destabilising cocharacters introduced by Hesselink, Kempf, and Rousseau.

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## 1. Introduction

The aim of this paper is to present a construction of the *semisimplification* of a subgroup  $H$  of a (possibly non-connected) reductive linear algebraic group  $G$  over an arbitrary field  $k$ . This construction unifies and generalises many concepts already in the literature within a single framework. For example, the semisimplification of a module for a group is a well-known construction in representation theory, corresponding in our case to the situation where  $H \subseteq \mathrm{GL}_n(k)$ . Building on this idea, for  $G$ , a connected

reductive linear algebraic group over a field  $k$ , and  $H$ , a subgroup of  $G(k)$ , Serre introduced the concept of a ‘ $G$ -analogue’ of semisimplification from representation theory in [19, Section 3.2.4]. This notion is also used for representations of various kinds of algebras: for example, see [12], [8], [16], [23], and [24]. It is also an ingredient in work of Lawrence-Sawin on the Shafarevich Conjecture for abelian varieties [13] and work of Lawrence-Venkatesh on Mordell’s Conjecture [14], which involve Galois representations taking values in possibly non-connected reductive  $p$ -adic groups.

We begin by recalling how the most basic case works. Let  $n \in \mathbb{N}$ , and let  $H$  be a subgroup of  $\mathrm{GL}_n(k)$ . There is an  $H$ -module filtration of  $k^n$  such that the successive quotients are irreducible, by the Jordan-Hölder Theorem. In terms of matrices, this implies that by changing basis if necessary, we may assume that  $H$  is in upper block-triangular form, with the action of  $H$  on each quotient being represented by the corresponding block on the diagonal. Letting  $H'$  be the subgroup of  $\mathrm{GL}_n(k)$  consisting of the block diagonal matrices obtained by taking each element of  $H$  and replacing the entries above the block diagonal with 0s, we obtain a subgroup that acts semisimply on  $k^n$ —that is,  $H'$  is completely reducible. Since this construction is independent of the choice of basis up to  $\mathrm{GL}_n(k)$ -conjugacy, again by the Jordan-Hölder Theorem, it is reasonable to call  $H'$  the *semisimplification* of  $H$ .

We now explain several of the ingredients of our construction in the case that  $k$  is algebraically closed, which removes some technicalities. Recall [2, 19] that if  $G$  is connected and  $H$  is a subgroup of  $G$ , then  $H$  is  *$G$ -completely reducible* ( $G$ -cr for short) if for any parabolic subgroup  $P$  of  $G$  such that  $P$  contains  $H$ , there is a Levi subgroup  $L$  of  $P$  such that  $L$  contains  $H$ . If  $G = \mathrm{GL}_n$ , then  $H$  is  $G$ -cr if and only if  $k^n$  is completely reducible as an  $H$ -module; this follows from the usual characterisation of parabolic subgroups of  $\mathrm{GL}_n$  as stabilizers of flags of subspaces. We make the same definition for arbitrary reductive  $G$ , replacing parabolic subgroups and Levi subgroups with R-parabolic subgroups and R-Levi subgroups instead (see Section 2 for details).

To perform our construction, we apply a characterisation of  $G$ -complete reducibility in terms of geometric invariant theory (GIT). We see this idea already in our original example: we can view  $H'$  as a degeneration of  $H$  in the following sense. Let the sizes of the blocks down the diagonal be  $n_1, \dots, n_r$ , and define a cocharacter  $\lambda: \mathbb{G}_m \rightarrow \mathrm{GL}_n$  by

$$\lambda(a) = \mathrm{diag}(a^{r'}, \dots, a^{r'}, \dots, a^1, \dots, a^1), \text{ with } n_i \text{ occurrences of } a^{r-i+1}, 1 \leq i \leq r.$$

For each  $a \in k^*$ , define  $H_a = \lambda(a)H\lambda(a)^{-1}$  for  $a \in k^*$ . Then  $H' = \lim_{a \rightarrow 0} H_a$  in an appropriate sense.

Our definition of  $k$ -semisimplification (Definition 4.1) for arbitrary  $k$  is new, generalizes the one given by Serre in [19, Section 3.2.4], and is closely related to the definition given in [6] using optimal destabilising cocharacters; the two notions agree whenever the latter makes sense (see also [15, Section 4] for the algebraically closed case). We prove that the  $k$ -semisimplification of a subgroup  $H$  of  $G$  is unique up to conjugacy (Theorem 4.5), generalizing [19, Proposition 3.3(b)]. In Theorem 5.4, we show that a normal subgroup of a  $G$ -completely reducible subgroup  $H$  is  $G$ -completely reducible and that the process of  $k$ -semisimplification behaves well under passing to normal subgroups of  $H$ , if  $k$  is perfect or  $G$  is connected. The proof rests on deep results from the theory of spherical buildings and the Hesselink-Kempf-Rousseau theory of optimal destabilising cocharacters. We give a short and self-contained exposition, bringing together some results (such as Corollary 3.5) that follow from previous work but are not easily extracted from earlier papers.

## 2. Cocharacter-closed orbits

Following [7] and our earlier work [6, 1], we regard an affine variety over a field  $k$  as a variety  $X$  over the algebraic closure  $\bar{k}$  together with a choice of  $k$ -structure. We denote the separable closure of  $k$  by  $k_s$ . We write  $X(k)$  for the set of  $k$ -points of  $X$  and  $X(\bar{k})$  (or just  $X$ ) for the set of  $\bar{k}$ -points of  $X$ . By a subvariety of  $X$ , we mean a closed  $\bar{k}$ -subvariety of  $X$ ; a  $k$ -subvariety is a subvariety that is defined over  $k$ . We denote by  $M_n$  the associative algebra of  $n \times n$  matrices over  $k$ . Below  $G$  denotes a possibly non-connected reductive linear algebraic group over  $k$ . By a subgroup of  $G$ , we mean a closed  $\bar{k}$ -subgroup;

and by a  $k$ -subgroup, we mean a subgroup that is defined over  $k$ . (We note here that much of what follows works for non-closed subgroups—most of the important conditions hold for  $H$  if and only if they hold for the Zariski closure  $\bar{H}$ ; the details are left to the reader.) By  $G^0$ , we denote the identity component of  $G$ , and likewise for subgroups of  $G$ .

We define  $Y_k(G)$  to be the set of  $k$ -defined cocharacters of  $G$  and  $Y(G) := Y_{\bar{k}}(G)$  to be the set of all cocharacters of  $G$ .

Let  $H$  be a subgroup of  $G$ . Even if  $H$  is  $k$ -defined, the (set-theoretic) centralizer  $C_G(H)$  need not be  $k$ -defined in general. It is useful to have criteria to ensure that  $C_G(H)$  is  $k$ -defined (see Proposition 3.4 and Section 5). For instance, if  $k$  is perfect and  $H$  is  $k$ -defined, then  $C_G(H)$  is  $k$ -defined. We say that  $H$  is *separable* if the scheme-theoretic centralizer  $\mathcal{C}_G(H)$  is smooth [2, Definition 3.27]; for instance, any subgroup of  $\mathrm{GL}_n$  is separable [2, Example 3.28] (see [5] for more examples of separable subgroups). If  $H$  is  $k$ -defined and separable, then  $C_G(H)$  is  $k$ -defined (see [1, Proposition 7.4]).

Next we recall some basic notation and facts concerning parabolic subgroups in (non-connected) reductive groups  $G$  from [2, Section 6] and [6]. Given  $\lambda \in Y(G)$ , we define

$$P_\lambda = \{g \in G \mid \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$$

and  $L_\lambda = C_G(\mathrm{Im}(\lambda))$  (for the definition of a limit, see [20, Section 3.2.13]). We call  $P_\lambda$  an *R-parabolic subgroup* of  $G$  and  $L_\lambda$  an *R-Levi subgroup* of  $P_\lambda$ ; they are subgroups of  $G$ . We have  $P_\lambda = L_\lambda = G$  if  $\mathrm{Im}(\lambda)$  is contained in the centre of  $G$ . For ease of reference, we record without proof some basic facts about these subgroups.

### Lemma 2.1.

- (i) If  $P$  is a  $k$ -defined R-parabolic subgroup, then  $R_u(P)$  is  $k$ -defined.
- (ii) If  $P$  is a parabolic subgroup of  $G^0$ , then the normalizer  $N_G(P)$  is an R-parabolic subgroup of  $G$ , and  $N_G(P)$  is  $k$ -defined if  $P$  is.

If  $G$  is connected, then every pair  $(P, L)$  consisting of a parabolic  $k$ -subgroup  $P$  of  $G$  and a Levi  $k$ -subgroup  $L$  of  $P$  is of the form  $(P, L) = (P_\lambda, L_\lambda)$  for some  $\lambda \in Y_k(G)$ , and vice versa [20, Lemma 15.1.2(ii)]. In general, if  $\lambda \in Y_k(G)$ , then  $P_\lambda$  and  $L_\lambda$  are  $k$ -defined [6, Lemma 2.5], but the converse is not so straightforward. If  $P$  is an R-parabolic  $k$ -subgroup and  $L$  is an R-Levi  $k$ -subgroup of  $P$ , then for any maximal  $k$ -torus  $T$  of  $L$ , there exists  $\lambda \in Y_{k_s}(T)$  such that  $P = P_\lambda$  and  $L = L_\lambda$ . However, it is possible that  $P$  is a  $k$ -defined R-parabolic subgroup and yet there does not exist any  $\mu \in Y_k(G)$  such that  $P = P_\mu$ , and similarly for R-Levi subgroups—see [6, Remark 2.4]. This complicates some of the arguments below.

**Lemma 2.2.** Let  $P$  be an R-parabolic subgroup of  $G$  and  $L$  an R-Levi subgroup of  $P$ .

- (i) We have  $P \cong L \rtimes R_u(P)$ , and this is a  $k$ -isomorphism if  $P$  and  $L$  are  $k$ -defined.
- (ii) Any two R-Levi  $k$ -subgroups of an R-parabolic  $k$ -subgroup  $P$  are  $R_u(P)(k)$ -conjugate.

We denote the canonical projection from  $P$  to  $L$  by  $c_L$ ; this is  $k$ -defined if  $P$  and  $L$  are. If we are given  $\lambda \in Y(G)$  such that  $P = P_\lambda$  and  $L = L_\lambda$ , then we often write  $c_\lambda$  instead of  $c_L$ . We have  $c_\lambda(g) = \lim_{a \rightarrow 0} \lambda(a)g\lambda(a)^{-1}$  for  $g \in P_\lambda$ ; the kernel of  $c_\lambda$  is the unipotent radical  $R_u(P_\lambda)$ , and the set of fixed points of  $c_\lambda$  is  $L_\lambda$ .

Let  $m \in \mathbb{N}$ . Below we consider the action of  $G$  on  $G^m$  by simultaneous conjugation:  $g \cdot (g_1, \dots, g_m) = (gg_1g^{-1}, \dots, gg_mg^{-1})$ . Given  $\lambda \in Y(G)$ , we have a map  $P_\lambda^m \rightarrow L_\lambda^m$  given by  $\mathbf{g} \mapsto \lim_{a \rightarrow 0} \lambda(a) \cdot \mathbf{g}$ ; we abuse notation slightly and also call this map  $c_\lambda$ . For any  $\mathbf{g} \in P_\lambda^m$ , there exists an R-Levi  $k$ -subgroup  $L$  of  $P_\lambda$  with  $\mathbf{g} \in L^m$  if and only if  $c_\lambda(\mathbf{g}) = u \cdot \mathbf{g}$  for some  $u \in R_u(P_\lambda)(k)$ .

Our main tool from GIT is the notion of cocharacter-closure, introduced in [6] and [1].

**Definition 2.3.** Let  $X$  be an affine  $G$ -variety and let  $x \in X$  (we do not require  $x$  to be a  $k$ -point). We say that the orbit  $G(k) \cdot x$  is cocharacter-closed over  $k$  if for all  $\lambda \in Y_k(G)$  such that  $x' := \lim_{a \rightarrow 0} \lambda(a) \cdot x$  exists,  $x'$  belongs to  $G(k) \cdot x$ . If  $k = \bar{k}$  then it follows from the Hilbert-Mumford Theorem that  $G(k) \cdot x$  is cocharacter-closed over  $k$  if and only if  $G(k) \cdot x$  is closed in  $X$ .

then we say that  $\mathcal{O}$  is accessible from  $x$  over  $k$  if there exists  $\lambda \in Y_k(G)$  such that  $x' := \lim_{a \rightarrow 0} \lambda(a) \cdot x$  belongs to  $\mathcal{O}$ .

**Example 2.4.** If  $X = G^m$ ,  $\lambda \in Y_k(G)$ , and  $\mathbf{g} \in P_\lambda^m$ , then  $G(k) \cdot c_\lambda(\mathbf{g})$  is accessible from  $\mathbf{g}$  over  $k$ .

The following result is [1, Theorem 1.3].

**Theorem 2.5 (Rational Hilbert-Mumford Theorem).** Let  $G$ ,  $X$ ,  $x$  be as above. Then there is a unique  $G(k)$ -orbit  $\mathcal{O}$  such that  $\mathcal{O}$  is cocharacter-closed over  $k$  and accessible from  $x$  over  $k$ .

### 3. $G$ -complete reducibility

**Definition 3.1.** Let  $H$  be a subgroup of  $G$ . We say that  $H$  is  $G$ -completely reducible over  $k$  ( $G$ -cr over  $k$ ) if for any  $R$ -parabolic  $k$ -subgroup  $P$  of  $G$  such that  $P$  contains  $H$ , there is an  $R$ -Levi  $k$ -subgroup  $L$  of  $P$  such that  $L$  contains  $H$ . We say that  $H$  is  $G$ -irreducible over  $k$  ( $G$ -ir over  $k$ ) if  $H$  is not contained in any proper  $R$ -parabolic  $k$ -subgroup of  $G$  at all.

**Remark 3.2.** We say that  $H$  is  $G$ -cr if  $H$  is  $G$ -cr over  $\bar{k}$ —cf. Section 1. More generally, if  $k'/k$  is an algebraic field extension, then we may regard  $G$  as a  $k'$ -group, and it makes sense to ask whether  $H$  is  $G$ -cr over  $k'$ .

For more on  $G$ -complete reducibility, see [18, 19, 2].

Note that the definitions make sense even if  $H$  is not  $k$ -defined. It is immediate that  $G$ -irreducibility over  $k$  implies  $G$ -complete reducibility over  $k$ . We have  $P_{g \cdot \lambda} = gP_\lambda g^{-1}$  and  $L_{g \cdot \lambda} = gL_\lambda g^{-1}$  for any  $\lambda \in Y(G)$  and any  $g \in G$  (see, for example, [2, Section 6]). It follows that if  $H$  is  $G$ -cr over  $k$  (respectively,  $G$ -ir over  $k$ ), then so is any  $G(k)$ -conjugate of  $H$ . More generally, one can show that if  $H$  is  $G$ -cr over  $k$  (respectively,  $G$ -ir over  $k$ ), then so is  $\phi(H)$  for any  $k$ -defined automorphism  $\phi$  of  $G$ . If  $k = \bar{k}$  and  $H$  is  $G$ -cr, then  $H$  is reductive [19, Proposition 4.1] and [2, Section 2.4, Section 6.2]. It follows from Proposition 3.4 below that if  $H$  is  $k$ -defined,  $k$  is perfect and  $H$  is  $G$ -cr over  $k$ , then  $H$  is reductive. We see below (Corollary 3.5) that the converse holds in characteristic 0. On the other hand, the converse is false in general, as is shown by the example in [22, Proof of Proposition 1.10].

We now explain the link between  $G$ -complete reducibility and GIT. Fix a  $k$ -embedding  $\iota: G \rightarrow \mathrm{GL}_n$  for some  $n \in \mathbb{N}$ . Let  $H$  be a subgroup of  $G$ . Let  $m \in \mathbb{N}$ , and let  $\mathbf{h} = (h_1, \dots, h_m) \in H^m$ . We call  $\mathbf{h}$  a *generic tuple for  $H$  with respect to  $\iota$*  if  $h_1, \dots, h_m$  generate the subalgebra of  $M_n$  generated by  $H$  [6, Definition 5.4]. Note that we don't insist that  $\mathbf{h}$  is a  $k$ -point. Our constructions below do not depend on the choice of  $\iota$ , so we suppress the words 'with respect to  $\iota$ '. It is immediate that if  $\mathbf{h} \in H^m$  is a generic tuple for  $H$  and  $g \in G$ , then  $g \cdot \mathbf{h}$  is a generic tuple for  $gHg^{-1}$ .

**Theorem 3.3 ([1, Theorem 9.3]).** Let  $H$  be a subgroup of  $G$ , and let  $\mathbf{h} \in H^m$  be a generic tuple for  $H$ . Then  $H$  is  $G$ -completely reducible over  $k$  if and only if  $G(k) \cdot \mathbf{h}$  is cocharacter-closed over  $k$ .

Using this result, one can derive many results on  $G$ -complete reducibility: for instance, see [2] for the algebraically closed case and [6, 1] for arbitrary  $k$ . Note that if  $\mathbf{h} \in H^m$  is a generic tuple for  $H$ , then the centralizer  $C_G(H)$  coincides with the stabilizer  $G_{\mathbf{h}}$ .

**Proposition 3.4.** Let  $H$  be a  $k$ -subgroup of  $G$ . Suppose  $k$  is perfect. Then  $H$  is  $G$ -completely reducible over  $k$  if and only if  $H$  is  $G$ -completely reducible.

*Proof.* If  $k$  is perfect, then  $\bar{k}/k$  is separable and  $C_G(H)$  is  $k$ -defined. The result now follows from [1, Corollary 9.7(i)].  $\square$

**Corollary 3.5.** Suppose  $\mathrm{char}(k) = 0$ . Let  $H$  be a  $k$ -subgroup of  $G$ . Then  $H$  is  $G$ -completely reducible over  $k$  if and only if  $H$  is reductive.

*Proof.* If  $k = \bar{k}$ , then this is well known (see [19, Proposition 4.2] and [2, Section 2.2, Section 6.3], for

Recall that if  $S$  is a  $k$ -split torus of  $G$ , then  $C_G(S)$  is an R-Levi  $k$ -subgroup of  $G$  [1, Lemma 2.5]. Part (i) of the next result gives the converse, and part (ii) strengthens [1, Corollary 9.7(ii)]: we do not need the hypotheses that  $H$  and  $C_G(H)$  are  $k$ -defined. See also [19, Proposition 3.2].

**Proposition 3.6.** *Let  $L$  be an R-Levi  $k$ -subgroup of  $G$ , and let  $H$  be a subgroup of  $L$ .*

- (a) *There exists a  $k$ -split torus  $S$  in  $G$  such that  $L = C_G(S)$ .*
- (b)  *$H$  is  $G$ -completely reducible over  $k$  if and only if  $H$  is  $L$ -completely reducible over  $k$ .*

*Proof.* (a). We can choose  $\lambda \in Y_{k_s}(G)$  such that  $L = C_G(\text{Im}(\lambda))$ . Let  $\lambda = \lambda_1, \lambda_2, \dots, \lambda_r \in Y_{k_s}(G)$  be the  $\text{Gal}(k_s/k)$ -conjugates of  $\lambda$ , and let  $S$  be the subtorus of  $Z(L)^0$  generated by the subtori  $\text{Im}(\lambda_i)$ . Then  $S$  is  $k$ -defined, and  $L = C_G(S)$ . The product map  $\lambda_1 \times \dots \times \lambda_r$  gives an epimorphism from  $\bar{k}^* \times \dots \times \bar{k}^*$  onto  $S$ . But a quotient of a split  $k$ -torus is  $k$ -split [7, Corollary III.8.4], so  $S$  is split.

(b). Given (a), the result now follows from Theorem 3.3 together with [1, Theorem 5.4(ii)].  $\square$

We finish the section with some results involving non-connected reductive groups that are needed in the sequel. Note that if  $Q$  is an R-parabolic  $k$ -subgroup of  $G$  and  $M$  is an R-Levi  $k$ -subgroup of  $Q$ , then  $Q^0$  is a parabolic  $k$ -subgroup of  $G^0$ , and  $M^0$  is a Levi  $k$ -subgroup of  $Q^0$ ; see [2, Section 6].

**Lemma 3.7.** *Let  $P$  be an R-parabolic subgroup of  $G$ , and let  $T$  be a maximal torus of  $P$ . Then there is a unique R-Levi subgroup  $L$  of  $P$  such that  $T \subseteq L$ . If  $P$  and  $T$  are  $k$ -defined, then  $L$  is  $k$ -defined.*

*Proof.* The first assertion is [2, Corollary 6.5]. For the second, suppose  $P$  and  $T$  are  $k$ -defined. Then the unique R-Levi subgroup  $L$  of  $P$  containing  $T$  must be Galois-stable and hence  $k$ -defined also.  $\square$

**Lemma 3.8.**

- (a) *Let  $Q$  be an R-parabolic  $k$ -subgroup of  $G$ , and set  $P = Q^0$ . Then the R-Levi  $k$ -subgroups of  $Q$  are precisely the subgroups of the form  $N_Q(L)$  for  $L$ , a Levi  $k$ -subgroup of  $P$ .*
- (b) *Let  $Q, P$  be as in (a), and let  $H$  be a subgroup of  $P$ . Then  $H$  is contained in an R-Levi  $k$ -subgroup of  $Q$  if and only if  $H$  is contained in a Levi  $k$ -subgroup of  $P$ . Moreover, if  $L$  is a Levi  $k$ -subgroup of  $P$ , then  $c_{N_Q(L)}(H)$  is  $N_Q(L)$ -completely reducible over  $k$  if and only if  $c_L(H)$  is  $L$ -completely reducible over  $k$ .*
- (c) *Let  $H$  be a subgroup of  $G^0$ . Then  $H$  is  $G$ -completely reducible over  $k$  if and only if  $H$  is  $G^0$ -completely reducible over  $k$ .*

*Proof.* (a) As observed above, if  $M$  is an R-Levi subgroup of  $Q$ , then  $M^0$  is a Levi subgroup of  $P$ , and  $N_Q(M^0)^0 = N_P(M^0)^0 = M^0$ . Let  $L$  be a Levi subgroup of  $P$ , and let  $T$  be a maximal torus of  $L$ . By Lemma 3.7 there is a unique R-Levi subgroup  $M$  of  $Q$  such that  $T \subseteq M$ . The Levi subgroups  $M^0$  and  $L$  of  $P$  both contain  $T$ , so by Lemma 3.7, they are equal; in particular,  $M$  normalizes  $L$ . Now  $N_Q(T)$  normalizes  $L$  by Lemma 3.7, so  $N_Q(L)$  meets every component of  $Q$ . Since  $Q = M \ltimes R_u(Q)$ ,  $M$  also meets every component of  $Q$ . It follows that  $M = N_Q(L)$ . Finally,  $L$  contains a maximal  $k$ -torus of  $P$  if and only if  $N_Q(L)$  does, so  $L$  is  $k$ -defined if and only if  $N_Q(L)$  is, by Lemma 3.7.

(b) The first assertion follows immediately from (a), and part (c) now follows. For the second assertion of (b), note that the restriction of  $c_{N_Q(L)}(H)$  to  $P$  is  $c_L(H)$ ; the desired result now follows from part (c) applied to the reductive  $k$ -group  $N_Q(L)$ .  $\square$

#### 4. $k$ -semisimplification

Now we come to our main definition.

**Definition 4.1.** *Let  $H$  be a subgroup of  $G$ . We say that a subgroup  $H'$  of  $G$  is a  $k$ -semisimplification of  $H$  (for  $G$ ) if there exist an R-parabolic  $k$ -subgroup  $P$  of  $G$  and an R-Levi  $k$ -subgroup  $L$  of  $P$  such that  $H \subseteq P$  and  $H' = c_L(H)$ , and  $H'$  is  $G$ -completely reducible (or equivalently, by Proposition 3.6(ii),  $L$ -completely reducible) over  $k$ . We say the pair  $(P, L)$  yields  $H'$ .*



**Remarks 4.2.**

- (a) Let  $H$  be a subgroup of  $G$ . If  $H$  is  $G$ -cr over  $k$ , then clearly  $H$  is a  $k$ -semisimplification of itself, yielded by the pair  $(G, G)$ .
- (b) Suppose  $(P, L)$  yields a  $k$ -semisimplification  $H'$  of  $H$ . Let  $L_1$  be another  $R$ -Levi  $k$ -subgroup of  $P$ . Then  $L_1 = uLu^{-1}$  for some  $u \in R_u(P)(k)$ , so  $c_{L_1}(H) = uc_L(H)u^{-1}$ . Hence  $(P, L_1)$  also yields a  $k$ -semisimplification of  $H$ . We say that  $P$  yields a  $k$ -semisimplification of  $H$ .
- (c) It is straightforward to check that if  $\phi$  is an automorphism of  $G$  (as a  $k$ -group),  $H$  is a subgroup of  $G$ ; and if  $(P, L)$  yields a  $k$ -semisimplification  $H'$  of  $H$ , then  $\phi(H')$  is a  $k$ -semisimplification of  $\phi(H)$ , yielded by  $(\phi(P), \phi(L))$ .
- (d) For  $G$  connected and  $H$  a subgroup of  $G(k)$ , Definition 4.1 recovers Serre's ' $G$ -analogue' of a semisimplification from [19, Section 3.2.4]. For  $k = \bar{k}$ , Definition 4.1 generalizes the definition of  $\mathcal{D}(H)$  following [15, Lemma 4.1].

**Remark 4.3.** Let  $\mathbf{h} = (h_1, \dots, h_m) \in H^m$  be a generic tuple for  $H$ . Note that  $c_\lambda$  extends in the obvious way to a homomorphism from a parabolic subalgebra  $\mathcal{P}_\lambda$  of  $M_n$  onto a Levi subalgebra  $\mathcal{L}_\lambda$  of  $\mathcal{P}_\lambda$ , and  $\mathcal{P}_\lambda$  contains the subalgebra  $\mathcal{A}$  generated by  $H$ . Since the elements  $h_i$  generate  $\mathcal{A}$ , the elements  $c_\lambda(h_i)$  generate  $c_\lambda(\mathcal{A})$ . But  $c_\lambda(\mathcal{A})$  is the subalgebra of  $\mathcal{L}_\lambda$  generated by  $c_\lambda(H)$ , so we deduce that  $c_\lambda(\mathbf{h}) = (c_\lambda(h_1), \dots, c_\lambda(h_m))$  is a generic tuple for  $c_\lambda(H)$ . Hence by Theorem 3.3,  $c_\lambda(H)$  is a  $k$ -semisimplification of  $H$  if and only if  $G(k) \cdot c_\lambda(\mathbf{h})$  is cocharacter-closed over  $k$ . It follows from Theorem 2.5 that  $H$  admits at least one  $k$ -semisimplification: for we can choose  $\lambda \in Y_k(G)$  such that  $G(k) \cdot c_\lambda(\mathbf{h})$  is cocharacter-closed over  $k$ , so  $c_\lambda(H)$  is a  $k$ -semisimplification of  $H$ , yielded by  $(P_\lambda, L_\lambda)$ .

**Lemma 4.4.** Suppose that  $H'$  is a  $k$ -semisimplification of  $H$ . Then there is  $\lambda \in Y_k(G)$  such that  $H'$  is yielded by the pair  $(P_\lambda, L_\lambda)$ .

*Proof.* Suppose  $H'$  is yielded by the pair  $(P, L)$ . By the discussion in Section 2, there exist a maximal  $k$ -torus  $T$  of  $L$  and  $\mu \in Y_{k_s}(T)$  such that  $P = P_\mu$  and  $L = L_\mu$ . Choose a finite Galois extension  $k'/k$  such that  $T$  splits over  $k'$ , and let  $\lambda = \sum_{\gamma \in \text{Gal}(k'/k)} \gamma \cdot \mu \in Y_k(T)$ . One checks easily that  $H \subseteq P_\lambda$  and  $c_\lambda|_H = c_\mu|_H$  (see also the proof of [6, Lemma 2.5(ii)]). Hence  $(P_\lambda, L_\lambda)$  also yields  $H'$ .  $\square$

Here is our main result, which was proved in the special case  $k = \bar{k}$  in [6, Proposition 5.14(i)]; see also [19, Proposition 3.3(b)]. The uniqueness asserted in Theorem 4.5 is akin to the theorem of Jordan–Hölder.

**Theorem 4.5.** Let  $H$  be a subgroup of  $G$ . Then any two  $k$ -semisimplifications of  $H$  are  $G(k)$ -conjugate.

*Proof.* Let  $H_1, H_2$  be  $k$ -semisimplifications of  $H$ . By Lemma 4.4, there exist  $\lambda_1, \lambda_2 \in Y_k(G)$  such that  $(P_{\lambda_1}, L_{\lambda_1})$  realizes  $H_1$  and  $(P_{\lambda_2}, L_{\lambda_2})$  realizes  $H_2$ . Let  $\mathbf{h} \in H^m$  be a generic tuple for  $H$ . Then  $c_{\lambda_i}(\mathbf{h})$  is a generic tuple for  $H_i$  for  $i = 1, 2$ , and each orbit  $G(k) \cdot c_{\lambda_i}(\mathbf{h})$  is cocharacter-closed over  $k$  and accessible from  $\mathbf{h}$  over  $k$  (Example 2.4). It follows from the uniqueness result in Theorem 2.5 that the closed subset  $C_{\mathbf{h}} := \{g \in G \mid g \cdot c_{\lambda_1}(\mathbf{h}) = c_{\lambda_2}(\mathbf{h})\}$  contains a  $k$ -point.

Pick  $g \in C_{\mathbf{h}}$ . If  $H_2 = gH_1g^{-1}$ , then we are done. Otherwise, there exists  $h \in H$  such that  $gc_{\lambda_1}(h)g^{-1} \notin H_2$  or  $g^{-1}c_{\lambda_2}(h)g \notin H_1$ . Without loss, assume the former. We can repeat the above argument, replacing  $\mathbf{h}$  with the generic tuple  $\mathbf{h}' := (\mathbf{h}, h) \in H^{m+1}$ ; note that  $C_{\mathbf{h}'}$  is properly contained in  $C_{\mathbf{h}}$ . The result now follows by a descending chain condition argument.  $\square$

**Definition 4.6.** We define  $\mathcal{D}_k(H)$  to be the set of  $G(k)$ -conjugates of any  $k$ -semisimplification of  $H$  (see also the discussion preceding [15, Theorem 1.4]). This is well-defined by Theorem 4.5.

**Example 4.7.** Let  $H$  be a subgroup of  $G$ . As noted in Remark 4.2(a), if  $H$  is  $G$ -cr over  $k$ , then  $H$  is a  $k$ -semisimplification of itself, yielded by the pair  $(G, G)$ . If  $H$  is a  $G$ -ir subgroup of  $G$ , then  $H$  is the only  $k$ -semisimplification of  $H$ : this shows that not every element of  $\mathcal{D}_k(H)$  need be a  $k$ -semisimplification of  $H$ . In a similar vein, if  $P$  and  $Q$  are arbitrary  $R$ -parabolic  $k$ -subgroups of  $G$  and  $Q \supseteq P$ , then it is easily seen that  $Q$  yields a  $k$ -semisimplification of  $P$  if and only if  $P^0 = Q^0$ .

**Example 4.8.** Let  $H$  be a subgroup of  $G$  and let  $P$  be minimal among the  $R$ -parabolic  $k$ -subgroups that contain  $H$ . Let  $L$  be an  $R$ -Levi  $k$ -subgroup of  $P$ . We claim that  $c_L(H)$  is  $L$ -ir over  $k$  (see also [19, Proposition 3.3(a)] and [2, Section 3]); it then follows from Proposition 3.6(ii) that  $c_L(H)$  is a  $k$ -semisimplification of  $H$ . Suppose  $c_L(H)$  is not  $L$ -ir: say,  $c_L(H) \subseteq Q$ , where  $Q$  is a proper  $R$ -parabolic  $k$ -subgroup of  $L$ . There exist a maximal  $k$ -torus  $T$  of  $Q$  and cocharacters  $\lambda, \mu \in Y_{k_s}(T)$  such that  $P = P_\lambda$ ,  $L = L_\lambda$ , and  $Q = P_\mu$ . Now  $H \subseteq QR_u(P) \subsetneq P$ , and clearly  $QR_u(P)$  is  $k$ -defined. But it is easily checked that  $QR_u(P) = P_{m\lambda+\mu}$  for suitably large  $m \in \mathbb{N}$  (cf. [2, Lemma 6.2(i)]), so  $QR_u(P)$  is an  $R$ -parabolic  $k$ -subgroup of  $G$ , contradicting the minimality of  $P$ . Conversely, if  $P$  is an  $R$ -parabolic  $k$ -subgroup with  $R$ -Levi  $k$ -subgroup  $L$  such that  $P \supseteq H$  and  $c_L(H)$  is  $L$ -ir over  $k$ , then a similar argument shows that  $P$  is minimal among the  $R$ -parabolic  $k$ -subgroups containing  $H$ . This proves the claim.

In particular, let  $G, H, \lambda$ , and  $H'$  be as in the  $\mathrm{GL}_n$  example in Section 1. Let  $P = P_\lambda$  be the parabolic subgroup of block upper triangular matrices with blocks of size  $n_1, \dots, n_r$  down the leading diagonal. Let  $L = L_\lambda$  be the subgroup of block diagonal matrices with blocks of size  $n_1, \dots, n_r$  down the leading diagonal. Since each  $n_i \times n_i$  block yields an irreducible representation of  $H' := c_\lambda(H)$ ,  $H'$  is  $L$ -ir over  $k$ , so  $P$  is minimal among the  $R$ -parabolic  $k$ -subgroups of  $G$  containing  $H$ ; hence  $H'$  is the  $k$ -semisimplification of  $H$  yielded by  $(P, L)$ .

**Example 4.9.** Suppose  $\mathrm{char}(k) = 0$ . Let  $H$  be a  $k$ -subgroup of  $G$ , and let  $P$  be an  $R$ -parabolic subgroup of  $G$  with  $R$ -Levi subgroup  $L$  such that  $P \supseteq H$ . Then Corollary 3.5 implies that  $c_L(H)$  is a  $k$ -semisimplification of  $H$  if and only if  $R_u(H) \subseteq R_u(P)$ .

**Remark 4.10.** Given a reductive  $k$ -group  $G$  and a subgroup  $H$  of  $G$ , we may (as in Remark 3.2) regard  $G$  as a  $\bar{k}$ -group by forgetting the  $k$ -structure, so it makes sense to consider the semisimplification (that is, the  $\bar{k}$ -semisimplification) of  $H$ . The reader is warned that it can happen that  $H$  is  $G$ -cr over  $k$  but not  $G$ -cr, or vice versa (see [2, Example 5.11] and [5, Example 7.22]), so there is no direct relation between the notions of  $k$ -semisimplification and semisimplification.

## 5. Optimality and normal subgroups

In Example 4.7, we observed that not every element of  $\mathcal{D}_k(H)$  need be a  $k$ -semisimplification of  $H$ . On the other hand, it can happen that  $H$  is contained in many different  $R$ -parabolic subgroups of  $G$ , and there may exist many conjugate, but different,  $k$ -semisimplifications. We now recall two constructions that give under some extra hypotheses a more canonical choice of  $R$ -parabolic subgroup yielding a  $k$ -semisimplification. They apply in particular when  $G = \mathrm{GL}_n$  (see Example 5.6); this does not seem to be well known even when  $k = \bar{k}$ .

**First construction:** Suppose  $G$  is connected,  $H$  is a subgroup of  $G$ , and  $H$  is not  $G$ -cr over  $k$ . We use the theory of spherical buildings (see [18, 19]) and the argument of [3, Proof of Theorem 1.1]. Recall that the spherical building  $\Delta_k(G)$  of  $G$  is a simplicial complex whose simplices are the parabolic  $k$ -subgroups of  $G$ , ordered by reverse inclusion (the proper  $k$ -parabolic subgroups correspond to the non-empty simplices). The apartments of  $\Delta_k(G)$  are the sets of all  $k$ -parabolic subgroups of  $G$  that contain a fixed maximal split  $k$ -torus  $S$  of  $G$ . The set  $\Sigma$  of parabolic  $k$ -subgroups  $P$  of  $G$  such that  $P \supseteq H$  is a convex subcomplex of  $\Delta_k(G)$ , and  $\Sigma$  is not completely reducible in the sense of [19, Section 2.2] because  $H$  is not  $G$ -cr over  $k$  (see [19, Section 3.2.1]). By the Tits Centre Conjecture—see, for example, [4, Section 2.6] and [19, Section 2.4] and the references therein— $\Sigma$  has a so-called ‘centre’: a proper parabolic  $k$ -subgroup  $P_c \in \Sigma$  such that  $P_c$  is fixed by any building automorphism of  $\Delta_k(G)$  that stabilizes  $\Sigma$ . In particular,  $P_c$  is stabilized by any  $k$ -automorphism of  $G$  that stabilizes  $H$ .

**Lemma 5.1.** Let  $G, H$ , and  $\Sigma$  be as above. Let  $P_c$  be a centre for  $\Sigma$  such that  $P_c$  is not properly contained in any other centre for  $\Sigma$ . Then  $P_c$  yields a  $k$ -semisimplification of  $H$ .

*Proof.* Let  $\Lambda$  be the set of  $k$ -parabolic subgroups  $Q$  of  $G$  such that  $Q \subseteq P_c$ . Fix a Levi  $k$ -subgroup  $L$  of  $P_c$ . We have an inclusion-preserving bijection  $\psi$  from  $\Lambda$  to  $\Delta_k(L)$  given by  $Q \mapsto Q \cap L$ , with inverse given by  $R \mapsto RR_u(P_c)$ . Let  $\Sigma_L$  be the subset of  $\Delta_k(L)$  consisting of all the  $k$ -parabolic subgroups of



$L$  that contain  $c_L(H)$ . It is clear that  $\psi(\Sigma \cap \Lambda) = \Sigma_L$ . If  $\phi$  is a building automorphism of  $\Delta_k(G)$  that fixes  $P_c$ , then  $\phi$  stabilizes  $\Lambda$ , and we get an automorphism  $\phi_L$  of  $\Delta_k(L)$  (as a simplicial complex) given by  $\phi_L(Q \cap L) = \phi(Q) \cap L$ ; moreover, if  $\phi$  stabilizes  $\Sigma$ , then  $\phi_L$  stabilizes  $\Sigma_L$ .

We claim that  $\phi_L$  is a building automorphism of  $\Delta_k(L)$ . It is enough to show that  $\phi_L$  maps apartments to apartments. Let  $S$  be a maximal split  $k$ -torus of  $L$  (and hence of  $G$ ). Since  $\phi$  is a building automorphism, there is a maximal split  $k$ -torus  $S'$  of  $G$  such that for every  $k$ -parabolic subgroup  $Q$  of  $G$  that contains  $S$ ,  $\phi(Q)$  contains  $S'$ . In particular,  $S' \subseteq P_c$  since  $\phi(P_c) = P_c$ . By Lemma 3.7, there is a  $k$ -Levi subgroup  $L'$  of  $P_c$  such that  $S' \subseteq L'$ . By Lemma 2.2(ii), there exists  $u \in R_u(P_c)(k)$  such that  $uS'u^{-1} \subseteq L$ . Let  $R \in \Delta_k(L)$  such that  $S \subseteq R$ : say,  $R = Q \cap L$  for  $Q \in \Lambda$ . Then  $S' \subseteq \phi(Q)$ . Since  $\phi(Q) \subseteq P_c$ ,  $R_u(\phi(Q))$  contains  $R_u(P_c)$ , so  $uS'u^{-1} \subseteq \phi(Q)$ . Hence  $uS'u^{-1} \subseteq \phi(Q) \cap L = \phi_L(R)$ . This proves the claim.

Now suppose  $P_c$  does not yield a  $k$ -semisimplification of  $H$ . Then  $c_L(H)$  is not  $L$ -cr over  $k$ . By the discussion before the lemma,  $\Sigma_L$  has a centre  $R \subsetneq L$ . We have  $R = Q \cap L$  for some  $Q \in \Lambda$  with  $Q \subsetneq P_c$ . But the results in the previous paragraph imply that  $Q$  is a centre for  $\Sigma$ , contradicting the minimality of  $P_c$ .  $\square$

**Second construction:** We allow  $G$  to be non-connected again. Suppose the following property holds for a subgroup  $H$  of  $G$ :

(\*) there exists an R-parabolic  $k$ -subgroup  $P$  of  $G$  such that  $H \subseteq P$  but  $H$  is not contained in any R-Levi subgroup—that is, any R-Levi  $\bar{k}$ -subgroup—of  $P$ .

This hypothesis implies in particular that  $H$  is not  $G$ -cr over  $k$ . The construction in [6, Section 5.2] then yields a canonical so-called ‘optimal destabilising’ R-parabolic  $k$ -subgroup  $P_{\text{opt}}$  of  $G$  such that  $H \subseteq P_{\text{opt}}$  but  $H$  is not contained in any R-Levi subgroup of  $P_{\text{opt}}$ . If  $k$  is perfect then  $P_{\text{opt}}$  yields both a  $\bar{k}$ -semisimplification of  $H$  and a  $k$ -semisimplification of  $H$  by [11, Theorem 4.2], but both can fail for general  $k$ . Moreover,  $P_{\text{opt}}$  is stabilized by any  $k$ -automorphism of  $G$  that stabilizes  $H$ ; in particular, if  $M$  is a  $k$ -subgroup of  $G$  that normalizes  $H$  then  $M(k)$  normalizes  $P_{\text{opt}}$ . See [6, Theorem 5.16] for details.

This construction rests on the notion of an “optimal destabilising cocharacter” due to work of Hesselink [10], Kempf [11] and Rousseau [17]. Roughly speaking, the idea is as follows. Take a generic tuple  $\mathbf{h} \in H^m$  for  $H$ . Choose  $\mathbf{g} \in G^m$  such that  $G(k) \cdot \mathbf{g}$  is accessible from  $\mathbf{h}$  over  $k$  and  $G(k) \cdot \mathbf{g}$  is cocharacter-closed over  $k$ . Set  $\mathcal{O}(\mathbf{h}) = G(\bar{k}) \cdot \mathbf{g}$ ; note that  $\mathcal{O}(\mathbf{h})$  is uniquely defined by Theorem 2.5. Roughly speaking, we define  $\lambda_{\text{opt}} \in Y_k(G)$  to be the cocharacter that takes  $\mathbf{h}$  into  $\mathcal{O}(\mathbf{h})$  as quickly as possible (in an appropriate sense), and we define  $P_{\text{opt}}$  to be  $P_{\lambda_{\text{opt}}}$ . (In fact, we need a slight variation—due to Hesselink—on this construction: rather than taking a single generic tuple  $\mathbf{h}$ , one considers the action of a cocharacter  $\lambda$  on all elements of  $H$  at once.) Note that  $P_{\text{opt}}$  is not uniquely determined (see [6, Remark 5.22]).

Now suppose that  $H$  is a subgroup of  $G$  such that  $C_G(H)$  is  $k$ -defined. One can show that if  $H$  is  $G$ -cr then  $H$  is  $G$ -cr over  $k$  (as previously noted, the converse is false). In fact, we prove a slightly stronger result: if  $H$  is not  $G$ -cr over  $k$  then hypothesis (\*) holds. To see this, choose a generic tuple  $\mathbf{h} \in H^m$ . We can find  $\lambda \in Y_k(G)$  such that  $(P_\lambda, L_\lambda)$  yields a  $k$ -semisimplification  $H'$  of  $H$ ; so  $G(k) \cdot c_\lambda(\mathbf{h})$  is cocharacter-closed over  $k$  but  $G(k) \cdot \mathbf{h}$  is not. If  $H$  is contained in an R-Levi  $\bar{k}$ -subgroup  $L$  of  $P_\lambda$  then  $c_\lambda(\mathbf{h}) = u \cdot \mathbf{h}$  for some  $u \in R_u(P_\lambda)$ . But then [1, Theorem 7.1] implies that  $c_\lambda(\mathbf{h}) = u_1 \cdot \mathbf{h}$  for some  $u_1 \in R_u(P_\lambda)(k)$ , so  $G(k) \cdot c_\lambda(\mathbf{h}) = G(k) \cdot \mathbf{h}$ , a contradiction.

**Remark 5.2.** Let  $M$  be a  $k$ -subgroup of  $G$  such that  $M$  normalizes  $H$ , and let  $P$  be the R-parabolic subgroup of  $G$  obtained from one of the constructions above. Then it is automatic that  $M(k)$  normalizes  $P$ . However, under the extra hypothesis that  $H$  is  $k$ -defined, we can in fact show that  $M \subseteq N_G(P)$ . To see this, one can first extend the field from  $k$  to  $k_s$  and then show that the R-parabolic subgroup obtained from either of the constructions is  $k$ -defined (cf. [3, Proof of Theorem 1.1] and [11, Section 4]), and hence coincides with  $P$ —this implies that  $M(k_s)$ , and hence  $M$ , normalizes  $P$ .

**Remark 5.3.** There are some limitations on the constructions given above. First, without the hypothesis that  $k$  is perfect, it can happen that the subgroup obtained from  $P_{\text{opt}}$  is not  $G$ -cr over  $k$ , and is

therefore not a  $k$ -semisimplification of  $H$ . (It is, however,  $G(\bar{k})$ -conjugate to a  $k$ -semisimplification of  $H$ .) Second, as yet there is no theory of optimal destabilising subgroups that holds for arbitrary fields—this means that we do not know how to define a version of  $P_{\text{opt}}$  for a subgroup  $H$  that is not  $G$ -cr over  $k$  if  $(*)$  does not hold. See [6, Section 1 and Example 5.21] for further discussion of this latter point.

By combining the two constructions above we obtain the following “Clifford theory” result, exploring the link between the semisimplification of a group and a normal subgroup. In the case  $k$  is algebraically closed, part (a) is [2, Theorem 3.10].

**Theorem 5.4.** *Let  $M$  be a  $k$ -subgroup of  $G$ , and let  $H$  be a normal  $k$ -subgroup of  $M$ . Suppose at least one of the following holds:*

- (i)  $k$  is perfect.
- (ii)  $G$  is connected.

Then:

- (a) *If  $M$  is  $G$ -completely reducible over  $k$ , then  $H$  is  $G$ -completely reducible over  $k$ .*
- (b) *There is an  $R$ -parabolic subgroup  $P$  of  $G$  such that  $M \subseteq P$  and  $P$  yields both a  $k$ -semisimplification of  $M$  and a  $k$ -semisimplification of  $H$ . In particular, there exist  $k$ -semisimplifications  $M'$  (respectively,  $H'$ ) of  $M$  (respectively, of  $H$ ) such that  $H'$  is normal in  $M'$ .*

*Proof.* Suppose  $H$  is not  $G$ -cr over  $k$ . Choose  $P = P_{\text{opt}}$  in case (i) and  $P = P_c$  in case (ii). Then  $M \subseteq N_G(P)$  by Remark 5.2. Since  $H$  is not contained in any  $R$ -Levi  $k$ -subgroup of  $P$ ,  $H$  is not contained in any  $R$ -Levi  $k$ -subgroup of  $N_G(P)$  (Lemma 3.8). Hence  $M$  is not contained in any  $R$ -Levi  $k$ -subgroup of  $N_G(P)$ . It follows that  $M$  is not  $G$ -cr over  $k$ . This proves part (a).

For (b), pick  $\lambda \in Y_k(G)$  such that  $(P_\lambda, L_\lambda)$  yields a semisimplification  $M' := c_\lambda(M)$  of  $M$ . Then  $c_\lambda(M)$  is  $G$ -cr over  $k$ , and  $c_\lambda(H)$  is normal in  $c_\lambda(M)$ . Now  $c_\lambda(M)$  and  $c_\lambda(H)$  satisfy the hypotheses of the theorem, so  $c_\lambda(H)$  is  $G$ -cr over  $k$  by (a). Hence  $(P_\lambda, L_\lambda)$  yields a semisimplification  $H' := c_\lambda(H)$  of  $H$  as well, and  $H'$  is normal in  $M'$ .  $\square$

**Remark 5.5.** *The hypothesis in part (ii) can be weakened: one only needs to assume that  $H \subseteq G^0$ . In order to make the proof go through, one needs to verify that the first construction above extends to this situation.*

**Example 5.6.** *Let  $H$  be a  $k$ -subgroup of  $G = \text{GL}_n$  such that  $H$  is not completely reducible over  $k$ . Since  $H$  is separable,  $C_G(H)$  is  $k$ -defined, so  $H$  is not  $G$ -completely reducible; we obtain a parabolic  $k$ -subgroup  $P_{\text{opt}}$  as above which yields a subgroup  $H'$ . We claim that  $H'$  is a  $k$ -semisimplification of  $H$ . For suppose  $H'$  is not  $G$ -cr over  $k$ . Choose  $\mathbf{h}, \mathbf{g}$  as above, and let  $\mathbf{h}' = c_{\lambda_{\text{opt}}}(\mathbf{h})$  (so that  $\mathbf{h}'$  is a generic tuple for  $H'$ ). Since  $C_G(H')$  is  $k$ -defined, hypothesis  $(*)$  holds, so we obtain an optimal cocharacter which takes  $\mathbf{h}'$  out of  $G \cdot \mathbf{h}' = \mathcal{O}(\mathbf{h})$  and into  $\mathcal{O}(\mathbf{h}')$ . But  $\mathbf{g}$  is accessible from  $\mathbf{h}'$  over  $k$  by [1, Theorem 4.3(ii)], so  $\mathcal{O}(\mathbf{h}') = \mathcal{O}(\mathbf{h})$ , a contradiction.*

*The parabolic subgroup  $P_{\text{opt}}$  is the stabilizer of some flag  $\mathcal{F}$  of subspaces of  $k^n$ , and  $\mathcal{F}$  does not admit a complementary  $H$ -stable flag of subspaces of  $k^n$ . By Remark 5.2,  $C_G(H)$  is a subgroup of  $P_{\text{opt}}$ —that is,  $C_G(H)$  stabilizes  $\mathcal{F}$ —and likewise the normalizer  $N_G(H)$  stabilizes  $\mathcal{F}$  if  $N_G(H)$  is  $k$ -defined. If  $k$  is perfect then  $N_G(H)$  is automatically  $k$ -defined but it need not be  $k$ -defined in general; see [9] for further discussion.*

**Remark 5.7.** *Hesselink gives an example [10, Example 8.5] of a subgroup  $H$  of an almost simple group  $G$  of type  $C_2$  such that  $P_{\text{opt}}$  is not a minimal centre for  $\Sigma$ , the subcomplex of the building  $\Delta_k(G)$  of  $G$  consisting of all parabolic subgroups of  $G$  that contain  $H$ . This shows that the two constructions above can yield different  $R$ -parabolic subgroups. Nevertheless, the corresponding  $k$ -semisimplifications of  $H$  are  $G(k)$ -conjugate, thanks to Theorem 4.5.*

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